Pion Production in SIMC

Dave Gaskell

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1 Introduction

SIMC has been shown to be a powerful tool in analyzing elastic scattering, $(e, e'p)$, and kaon production data. This report will summarize how we extend SIMC to handle pion production and point out some of the ambiguities one must face in extending models of pion production from hydrogen to nuclei with $A > 1$.

2 Event Generation

In SIMC, the beam energy is fixed (modulo some small smearing) and the scattered electron quantities (E_e,$\cos\theta_e$,ϕ_e) are thrown flat in the lab frame. For pion production, the pion angles are also thrown flat. In the case of production from the free proton, the magnitude of the pion momentum is fixed by energy and momentum conservation. In the case of production from the Deuteron and Helium-3 the following procedure is followed. It is assumed that the pion is produced from a single nucleon in the nucleus and the other nucleon(s) are non-participating spectators. A momentum, $\vec{p}_{fermi}$, is chosen for the target nucleon. The magnitude of the nucleon momentum is drawn from a normalized distribution derived from an appropriate momentum space wave function, while the direction is thrown flat in ϕ and $cos\theta$. The final element that must be fixed is the target nucleon energy. In general, the nucleons in a nucleus are not on mass shell, so the nucleon momentum does not fix its energy. For deuterium, we choose the target nucleon energy to be defined as

$$E_{target} = M_D - \sqrt{m^2_{spectator} + p^2_{fermi}}\,.$$

This just puts the spectator on mass shell with the energy of the struck nucleon given by energy-momentum conservation.

In the case of helium-3, the off-shell prescription is more complicated. Not only is there the momentum of the struck nucleon, but the two spectators can have some relative momentum between them. To address this issue, a spectral function is used. A spectral function describes the probability to find a nucleon in a nucleus with some momentum and energy. Typically, this is described in terms of the so-called missing momentum, P_m, and missing energy, E_m. In this case, P_m is simply what we have called p_{Fermi}, while E_m is the difference between the invariant mass of the spectator nucleons

+ (on-shell) struck nucleon mass from the target nucleus mass,

$$E_m = M_{2spec} + M_{struck} - M_{3He} \, .$$

In our case, we use E_m from the spectral function to determine the invariant mass of the 2-spectator system. Then the energy of the struck nucleon is given by,

$$E_{struck} = M_{3He} - \sqrt{M_{2spec}^2 + p_{Fermi}^2} \, .$$

The spectral function used is a factorized (we assume the E_m and P_m distributions are independent) approximation of a calculation [1] fit to ^{3}He$(e, e'p)$ data [2]. Note that the helium-3 prescription can be applied to any nucleus for which the appropriate spectral function is available.

Now with the target nucleon energy and momentum fixed, one can solve for the magnitude of the pion momentum using energy and momentum conservation. More specifically,

$$\omega + E_{target} = E_\pi + E_{recoil}$$

$$\vec{q} + \vec{p}_{fermi} = \vec{p}_\pi + \vec{p}_{recoil}.$$

Note that it is assumed that the recoiling struck nucleon is now on the mass shell (i.e. $E_{recoil}^2 = m_{recoil}^2 + p_{recoil}^2$).

3 Physics Model

3.1 Proton Target

In SIMC, each event is weighted by some model cross section calculated from the generated (vertex) kinematics. In the models used in SIMC, the electroproduction cross section is written as,

$$\frac{d\sigma}{d\Omega_e dE_e d\Omega_\pi^*} = \Gamma_v \frac{d\sigma}{d\Omega_\pi^*}$$

where Γ_v is the virtual photon flux defined as

$$\Gamma_v = \frac{\alpha}{2\pi^2} \frac{E'}{E} \frac{K_{eq}}{Q^2} \frac{1}{1-\epsilon}$$

and $\frac{d\sigma}{d\Omega^*}$ is the center of mass virtual photon cross section. K_{eq} is the equivalent photon energy (the energy required for a real photon to excite a nucleon to a mass W) and is defined (Hand convention)

$$K_{eq} = \frac{W^2 - m_n^2}{2m_n},$$

where m_n is the mass of the nucleon.

The center of mass virtual photon cross section is written in terms of transverse, longitudinal, and interference terms,

$$\frac{d\sigma}{d\Omega_\pi^*} = \frac{d\sigma_T}{d\Omega_\pi^*} + \epsilon \frac{d\sigma_L}{d\Omega_\pi^*} + \sqrt{2\epsilon(\epsilon+1)} \frac{d\sigma_{TL}}{d\Omega_\pi^*} \cos\phi_\pi^* + \epsilon \frac{d\sigma_{TT}}{d\Omega_\pi^*} \cos 2\phi_\pi^*.$$

The center of mass cross section is calculated differently depending on the model. For the high W data (W=1.6 GeV), we use Henk Blok's parameterization of $d\sigma/dtd\phi$ from Brauel et. al. [3], as well as the so-called MAID model. MAID is somewhat of a black box since we do not have the original code, although the paper describing the physics on which it is based is available [4]. In essence, the MAID model calculates resonant and non-resonant multipoles constrained by existing data. The MAID model is also used for the low W data (W=1.16 GeV) in addition to an older multipole model from von Gehlen [5] (which I will call Multipole). While MAID takes W, Q^2, and $\cos\theta_\pi^*$ as inputs and uses some analytical parameterization to calculate the multipoles, Multipole uses a simple look-up table (binned in K_{eq} and Q^2). In MAID and Multipole one ends up with response functions (R_T,R_L,etc.) that must be multiplied by a kinematic factor to get the center of mass cross sections. Modulo factors of two, the kinematic factors are identical for both models:

$$\frac{d\sigma_T}{d\Omega^*} = \frac{p_\pi W}{2K_{eq}m_n}R_T$$

$$\frac{d\sigma_L}{d\Omega^*} = \frac{p_\pi W}{K_{eq}m_n}R_L$$

$$\frac{d\sigma_{TL}}{d\Omega^*} = 2\frac{p_\pi W}{K_{eq}m_n}\mathrm{REAL}(R_{TL})$$

$$\frac{d\sigma_{TT}}{d\Omega^*} = \frac{p_\pi W}{2K_{eq}m_n}R_{TT}.$$

For pion production from the proton, the above prescription is unambiguous. Given an event, one simply needs to calculate Γ_v in the lab, boost the pion to the center of mass and calculate $d\sigma/d\Omega_\pi^*$. Then one must transform the virtual photon cross section back to the lab via a Jacobian (see Appendix A).

3.2 A>1

When A is greater than one, we take the approach that the model cross section is just the proton cross section with the caveat that the proton (or neutron) is now allowed to move. Implementing this alone is difficult enough, but we must also face the ambiguities inherent in extending the model to a particle that is not on the mass shell.

First we must deal with the virtual gamma flux (Γ_v). In calculating any cross section one generally starts with the inverse of the relative electron-target flux, v. Typically (i.e. for a stationary target) this is just E_e/p_e. When the target is allowed to move, however, we must account for the relative velocity of the target particle. Thus we must correct the incident flux, v, by a factor $[1 - \vec{p}_{fermi}\cdot\vec{p}_e/(E_{target}p_e)]$. Furthermore, if one takes K_{eq} to be the real photon energy needed to excite a mass W from the target nucleon we then might naively calculate,

$$K_{eq} = \frac{W^2 - (E_{target}^2 - p_{fermi}^2)}{2(E_{target} - \vec{p}_{fermi}\cdot\vec{q}/|\vec{q}|)}.$$

K_{eq} in the virtual photon flux cancels when combined with the K_{eq} in the kinematic factors for the center of mass photon cross section. However, since it is used in the Multipole model look-up table, the choice of convention is still important. It turns out that when the above form of K_{eq} is used in the Multipole model, the center of mass cross section does not have the correct dependence on W. For that reason, the same form for K_{eq} as was used in the stationary nucleon case is used in the moving nucleon case (this gives the correct W dependence). Note that, regardless of model, W is calculated,

$$W^2 = (E_{target} + \omega)^2 - (\vec{p}_{fermi} + \vec{q})^2.$$

A more subtle issue is how to treat the "off-shellness" of the target nucleon. Recall that in the generation of an event, the spectator system is on the mass shell while the target is not. However, the models we use generally assume the target is on mass shell. We are free to redefine the target nucleon energy such that the particle is on shell, but then this energy will be inconsistent with the energy calculated for the pion. More specifically, when the pion is boosted to the center of mass, it will not actually be boosted to a frame where the sum of all momenta is zero. Currently, we choose to treat the struck nucleon the same in both event generation and in the physics models, i.e. assume the strcuk nucleon is off-shell.

Once we choose a convention for the target nucleon energy, it is not always possible to implement it in a consistent manner. Both MAID and Multipole require that the fundamental amplitudes (squared) be multiplied by various kinematic factors. Since these are calculated outside the model, we are free to choose whatever convention we like. The amplitudes themselves, however, are another story. For the Multipole model, once one has chosen a prescription for calculating the "off-shell" K_{eq}, there are no ambiguities. The MAID model is a bit more complicated, though. Recall that the MAID model takes W, Q^2, and $\cos\theta^*$ as its inputs. It is unknown how it determines the fundamental amplitudes, but from the paper describing this model it almost certainly requires the center of mass virtual photon energy and momentum. The problem arises from the fact that we have calculated W and $\cos\theta^*$ assuming some off-shell prescription. When MAID determines ω^* from the input quantities, it assumes the target nucleon is on shell so that the virtual photon energy (and momentum), and therefore the amplitudes it returns may not be consistent with the kinematics that we generated for this event. This is not a problem if we change the event generation scheme and put the target nucleon on shell and put the spectator off shell, but then at some kinematic settings, the agreement between data and SIMC suffers in regard to distribution offsets. In addition, the spectator nucleon is now off shell in the final state. In the end, we have decided to tolerate the possible inconsistency in MAID's calculation of ω^* and use the "off-shell" event generation scheme described previously and take the corresponding values of W and $\cos\theta^*$ as inputs to the MAID model.

4 Appendix A - The Jacobian

In calculating the cross section section for a given event in SIMC, we generally use models that describe the virtual photon cross section in the center of mass ($d\sigma/d\Omega^*$) multiplied by a virtual photon flux factor (Γ_v). Thus, we must transform the virtual photon cross section to the lab in order to properly weight the event. Rather than do the transformation $d\sigma/d\Omega^*_\pi \to d\sigma/d\Omega_\pi$ all at once, it is somewhat simpler to do it in two steps. The first step is to transform $d\sigma/d\Omega^*_\pi$ to $d\sigma/dtd\phi^*_{q\pi}$. The invariant t is written,

$$t = -Q^2 + m_\pi^2 - 2\omega E_\pi + 2p_\pi q cos\theta_{q\pi} \;, \tag{1}$$

where all the relevant angles and momenta can be written in terms of lab or center of mass variables. In the center of mass, all energy and momenta are independent of the pion direction, $\theta_{q\pi}$, so

$$\frac{dt}{d\cos\theta^*_{q\pi}} = 2p^*_\pi q^* \;, \tag{2}$$

and

$$\frac{d\sigma}{dtd\phi^*_{q\pi}} = \frac{d\sigma}{d\Omega^*}\frac{d\cos\theta^*_{q\pi}}{dt} = \frac{1}{2p^*_\pi q^*}\frac{d\sigma}{d\Omega^*}. \tag{3}$$

The full transformation is then,

$$\frac{d\sigma}{d\Omega_\pi} = J(t,\phi^*_{q\pi} \to \cos\theta_{q\pi},\phi_{q\pi})\frac{1}{2p^*_\pi q^*}\frac{d\sigma}{d\Omega^*_\pi} \;, \tag{4}$$

where the Jacobian, J, is given by the determinant,

$$J(t,\phi^*_{q\pi} \to \cos\theta_{q\pi},\phi_{q\pi}) = \begin{vmatrix} \frac{\partial t}{\partial\cos\theta_{q\pi}} & \frac{\partial t}{\partial\phi_{q\pi}} \\ \frac{\partial\phi^*_{q\pi}}{\partial\cos\theta_{q\pi}} & \frac{\partial\phi^*_{q\pi}}{\partial\phi_{q\pi}} \end{vmatrix}. \tag{5}$$

Before proceeding with the calculation of the above determinant, it is worth noting a couple of useful derivatives. One derivative that will appear repeatedly is $\partial p_\pi/\partial\cos\theta_{q\pi}$.

$$\frac{\partial p_\pi}{\partial\cos\theta_{q\pi}} = \tag{6}$$

$$\frac{p_\pi p_{recoil}(\hat{\mathbf{q}}) + p_\pi^2\cos\theta_{q\pi} - p_\pi p_N \sin\theta_{qN}\frac{\cos\theta_{q\pi}}{\sin\theta_{q\pi}}F(\phi_{qN},\phi_{q\pi})}{p_\pi\sin^2\theta_{q\pi} - p_N\sin\theta_{qN}\sin\theta_{q\pi}F(\phi_{qN},\phi_{q\pi}) + E_{recoil}\frac{p_\pi}{E_\pi} - p_{recoil}(\hat{\mathbf{q}})\cos\theta q\pi} \;,$$

where $F(\phi_{qN},\phi_{q\pi}) = \cos\phi_{q\pi}\cos\phi_{qN} + \sin\phi_{q\pi}\sin\phi_{qN}$, E_{recoil} is the energy of the recoiling nucleon ($= \omega + E_N - E_\pi$), and $p_{recoil}(\hat{\mathbf{q}})$ is the component of the recoiling nucleon momentum parallel to $\mathbf{q}$ ($= q + p_N\cos\theta_{qN} - p_\pi\cos\theta_{q\pi}$). Another useful derivative is $\partial p_\pi/\partial\phi_{q\pi}$.

$$\frac{\partial p_\pi}{\partial\phi_{q\pi}} = \tag{7}$$

$$\frac{p_N\sin\theta_{qN}p_\pi\sin\theta_{q\pi}G(\phi_{qN},\phi_{q\pi})}{p_{recoil}(\hat{\mathbf{q}})\cos\theta_{q\pi} + p_N\sin\theta_{qN}\sin\theta_{q\pi}F(\phi_{qN},\phi_{q\pi}) - p_\pi\sin^2\theta_{q\pi} - E_{recoil}\frac{p_\pi}{E_\pi}} \;,$$

where $G(\phi_{qN},\phi_{q\pi}) = \cos\phi_{qN}\sin\phi_{q\pi} - \sin\phi_{qN}\cos\phi_{q\pi}$.

With these two partial derivatives in hand, the derivatives of t with respect to the pion angles are relatively simple:

$$\frac{\partial t}{\partial\cos\theta_{q\pi}} = 2p_\pi q + 2(q\cos\theta_{q\pi} - \omega\frac{p_\pi}{E_\pi})\frac{\partial p_\pi}{\partial\cos\theta_{q\pi}}, \tag{8}$$

$$\frac{\partial t}{\partial\phi_{q\pi}} = 2(q\cos\theta_{q\pi} - \omega\frac{p_\pi}{E_\pi})\frac{\partial p_\pi}{\partial\phi_{q\pi}} \;. \tag{9}$$

The derivatives of $\phi^*_{q\pi}$ are somewhat involved. The first step is to describe how $\phi_{q\pi}$ is defined in the lab. First, all relevant vectors are described in the lab using what will be called the "q" coordinate system. This coordinate system is defined such that the z-axis lies along $\mathbf{q}$. Then, since $\mathbf{q}$ and $\mathbf{P_{beam}}$ describe the scattering plane, the y-axis chosen to be in the direction $\mathbf{q} \times \mathbf{P_{beam}}$, and the x-axis is $\mathbf{y} \times \mathbf{z}$. Then $\phi_{q\pi}$ is simply $\arctan \frac{p_y}{p_x}$. Typically, when the target is stationary, the boost to the center of mass is along $\mathbf{q}$, so that $\phi_{q\pi}$ is the same in the lab and the center of mass since the boost does not affect transverse components of the pion momentum. When the target is allowed to move, however, the boost is not always along $\mathbf{q}$, so $\phi^*_{q\pi}$ must be recalculated.

To calculate $\phi^*_{q\pi}$, the "q" system is defined as above in terms of $\mathbf{q}$ and $\mathbf{P_{beam}}$, but evaluated in the center of mass. For clarity, the transformation will be described. Starting with some $\mathbf{p}_\pi$ in the lab "q" system, the pion momentum must be boosted to the center of mass (from this point the π subscript will be dropped - unless specifically noted, all momenta refer to the pion). In this calculation, the explicit angular dependence of p_x $(= p \sin\theta \cos\phi)$, etc. will be be fully written out so that it is clearer how to take the partial derivatives later:

$$
\begin{aligned}
p^*_x &= p \sin\theta_{q\pi} \cos\phi_{q\pi} + p\frac{\gamma - 1}{\beta^2}\beta_x(\beta_x \sin\theta_{q\pi}\cos\phi_{q\pi} + \beta_y \sin\theta_{q\pi}\sin\phi_{q\pi} + \beta_z \cos\theta_{q\pi}) \\
&\quad -\gamma\beta_x E\,, \qquad (10) \\
p^*_y &= p \sin\theta_{q\pi} \sin\phi_{q\pi} + p\frac{\gamma - 1}{\beta^2}\beta_y(\beta_x \sin\theta_{q\pi}\cos\phi_{q\pi} + \beta_y \sin\theta_{q\pi}\sin\phi_{q\pi} + \beta_z \cos\theta_{q\pi}) \\
&\quad -\gamma\beta_y E\,, \qquad (11) \\
p^*_z &= p \cos\theta_{q\pi} + p\frac{\gamma - 1}{\beta^2}\beta_z(\beta_x \sin\theta_{q\pi}\cos\phi_{q\pi} + \beta_y \sin\theta_{q\pi}\sin\phi_{q\pi} + \beta_z \cos\theta_{q\pi}) \\
&\quad -\gamma\beta_z E\,. \qquad (12)
\end{aligned}
$$

Note that the above quantities describe the components of the center of mass momentum in terms of the lab "q" coordinate system and must be rotated to the center of mass "q" coordinate system. This rotation is totally determined by the boosted beam and q vectors (so does not depend on the pion angles):

$$
\begin{aligned}
p^{*'}_x &= p^*_x R_{xx} + p^*_y R_{xy} + p^*_z R_{xz}\,, \qquad (13) \\
p^{*'}_y &= p^*_x R_{yx} + p^*_y R_{yy} + p^*_z R_{yz}\,. \qquad (14)
\end{aligned}
$$

Which completes the calculation of $\phi^*_{q\pi} = \arctan \frac{p^{*'}_y}{p^{*'}_x}$.

Now, at last the partial derivatives for the Jacobian can be calculated. First making use of the fact that $\tan\phi^*_{q\pi} = \frac{p^{*'}_y}{p^{*'}_x}$ and,

$$\frac{\partial \tan\phi^*_{q\pi}}{\partial X} = (1 + \tan^2\phi^*_{q\pi})\frac{\partial \phi^*_{q\pi}}{\partial X}, \qquad (15)$$

gives,

$$\frac{\partial \phi^*_{q\pi}}{\partial X} = \frac{p^{*'}_x \frac{\partial p^{*'}_y}{\partial X} - p^{*'}_y \frac{\partial p^{*'}_x}{\partial X}}{(p^{*'}_x)^2 + (p^{*'}_y)^2}\,, \qquad (16)$$

where X is either $\cos\theta_{q\pi}$ or $\phi_{q\pi}$. So, the derivative of $\phi^*_{q\pi}$ with respect to some lab variable can be done using the above formula, with the expressions for the $p^{*'}_n$ derived above and assuming that the R_{nn} and β_n do not depend on the pion angles (which they should not because they are calculated completely from initial quantities). The expression for the derivative of $\phi^*_{q\pi}$ involves derivatives of the components of the rotated, center of mass pion momentum. These derivatives are cumbersome and will not all be given here, but an example follows. Consider the derivative of the y-component of the boosted, rotated pion momentum with respect to lab $\phi_{q\pi}$,

$$\frac{\partial p_y^{*'}}{\partial \phi_{q\pi}} = R_{yx}\frac{\partial p_x^*}{\partial \phi_{q\pi}} + R_{yy}\frac{\partial p_y^*}{\partial \phi_{q\pi}} + R_{yz}\frac{\partial p_z^*}{\partial \phi_{q\pi}}. \tag{17}$$

The partial derivatives of the unrotated, boosted components are given by,

$$\begin{aligned}
\frac{\partial p_x^*}{\partial \phi_{q\pi}} &= -p\sin\theta_{q\pi}\sin\phi_{q\pi} + p\frac{\gamma-1}{\beta^2}\beta_x\sin\theta_{q\pi}(\beta_y\cos\phi_{q\pi} - \beta_x\sin\phi_{q\pi}) + \\
&\quad \left(\frac{p_x^* + \gamma\beta_x E}{p} - \gamma\beta_x\frac{p}{E}\right)\frac{\partial p}{\partial \phi_{q\pi}}, \qquad (18)\\
\frac{\partial p_y^*}{\partial \phi_{q\pi}} &= p\sin\theta_{q\pi}\cos\phi_{q\pi} + p\frac{\gamma-1}{\beta^2}\beta_y\sin\theta_{q\pi}(\beta_y\cos\phi_{q\pi} - \beta_x\sin\phi_{q\pi}) + \\
&\quad \left(\frac{p_y^* + \gamma\beta_y E}{p} - \gamma\beta_y\frac{p}{E}\right)\frac{\partial p}{\partial \phi_{q\pi}}, \qquad (19)\\
\frac{\partial p_z^*}{\partial \phi_{q\pi}} &= p\frac{\gamma-1}{\beta^2}\beta_z\sin\theta_{q\pi}(\beta_y\cos\phi_{q\pi} - \beta_x\sin\phi_{q\pi}) + \\
&\quad \left(\frac{p_z^* + \gamma\beta_x E}{p} - \gamma\beta_z\frac{p}{E}\right)\frac{\partial p}{\partial \phi_{q\pi}}. \qquad (20)
\end{aligned}$$

Although the algebra involved in the center of mass to lab transformation is tedious, it is straightforward. The above formulation was checked to make sure that it reduced to the simpler "collinear boost" case. An alternative, more elegant formulation can be found in Ref. [6].

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