Notes

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I. ASYMMETRY FORMULA

$$A_1 = \frac{1}{D(1+\eta\xi)}\Big[A_{180}\big(1+\chi\cot\alpha\big) + A_{80}\big(\chi\csc\alpha\big)\Big] \tag{1}$$

$$A_2 = \frac{\xi - (\chi/\eta)\cot\alpha}{D(1+\eta\xi)}\Big[A_{180} + A_\alpha\left(\frac{1}{\cos\alpha - \chi\sin\alpha\cos^2\phi}\right)\Big] \tag{2}$$

II. MATRIX ELEMENT DEFINITIONS

A. Kodaira, et.al. (1979) [1–3]

$$\int dx x^{n-1} g_1(x,Q^2) = \sum_i \frac{1}{2} a^i_n E^n_{1,i}(Q^2,g)$$

$$\int dx x^{n-1} g_2(x,Q^2) = -\frac{n-1}{2n}\sum_i \left[a^i_n E^n_{1,i}(Q^2,g) - d^i_n E^n_{2,i}(Q^2,g)\right]$$

for $n = 1,3,5\dots$ and $n = 3,5,\dots$.

B. Matsuda and Uematsu (1980) [4]

Target Mass Effects in Polarized Electroproduction. The Nachtmann moments in the massless limit are given as:

$$\begin{aligned} M^n_1 &\equiv \frac{1}{2} a_n E^n_1(Q^2,g) \\ &= \int dx x^{n-1} g_1(x,Q^2) \\ M^n_2 &\equiv \frac{1}{2} d_n E^n_2(Q^2,g) \\ &= \int dx x^{n-1}\big[g_1(x,Q^2) + \frac{n}{n-1} g_2(x,Q^2)\big] \end{aligned}$$

Here the a_n and d_n are off by the factor of two. To be consistent with Dong (2008) the following replacements need to be made:

$$a_n \to \tilde{a}_{n-1} \tag{3}$$

$$d_n \to \tilde{d}_{n-1} \tag{4}$$

with the assumption that $E^n_i(Q^2,0) = 1$.

C. Jaffe and Ji (1991) [5]

Jaffe and Ji have a slightly different convention for defining the matrix elements

$$\begin{aligned}\int x^n g_1(x,Q^2)dx &= \sum_i \frac{1}{2} a_n^i(\mu^2) F_{1,i}^n(Q^2,\mu^2) \\ \int x^n g_2(x,Q^2)dx &= -\frac{n}{2(n+1)} \sum_i \left[a_n^i(\mu^2) F_{1,i}^n(Q^2,\mu^2) - d_n^i F_{2,i}^n(Q^2,\mu^2)\right]\end{aligned} \tag{5}$$

for $n = 0, 2, 4\ldots$ and $n = 2, 4, \ldots$.

Shifting to $n \to n+1$ gives the result

$$\begin{aligned}\int x^{n-1} g_1(x,Q^2)dx &= \sum_i \frac{1}{2} a_{n-1}^i(\mu^2) F_{1,i}^{n-1}(Q^2,\mu^2) \\ \int x^{n-1} g_2(x,Q^2)dx &= -\frac{n-1}{2n} \sum_i \left[a_{n-1}^i(\mu^2) F_{1,i}^n(Q^2,\mu^2) - d_{n-1}^i(\mu^2) F_{2,i}^{n-1}(Q^2,\mu^2)\right]\end{aligned} \tag{6}$$

for $n = 1, 3, 5\ldots$ and $n = 3, 5, \ldots$, respectively. The matrix elements are then related to the CN projections defined below:

$$\sum_i a_{n-1}^i(\mu^2) = \tilde{a}_{n-1}(\mu^2) \qquad \text{for n=1,3,5,...} \tag{7}$$

$$\sum_i d_{n-1}^i(\mu^2) = \tilde{d}_{n-1}(\mu^2) \qquad \text{for n=3,5,...} \tag{8}$$

D. Piccone and Ridolfi (1997) [6]

This paper is on target mass corrections, but if we take $m \simeq 0$ (no TMCS) we get the following:

$$\begin{aligned}\int dx x^{n-1} g_1(x,Q^2) &= a_n + \mathcal{O}\left(\frac{m^2}{Q^2}\right) \\ \int dx x^{n-1} g_2(x,Q^2) &= \frac{n-1}{n}(d_n - a_n) + \mathcal{O}\left(\frac{m^2}{Q^2}\right)\end{aligned}$$

for $n = 1, 3, 5\ldots$ and $n = 3, 5, \ldots$.

E. Blumlein and Tkabladze (1999)[?]

$$g_{10}^n = a_n, \qquad \text{for } n = 0, 2... \tag{9}$$

$$g_{20}^n = \frac{n}{n+1}(d_n - a_n), \qquad \text{for } n = 2, 4... \tag{10}$$

where the moments are defined as

$$g_{i0}^{n} = \int x^{n} g_i(x) dx \tag{11}$$

Therefore taking $n \to n-1$ we get

$$g_{10}^{n} = a_{n-1}, \qquad \text{for } n = 1, 3... \tag{12}$$

$$g_{20}^{n} = \frac{n-1}{n}(d_{n-1} - a_{n-1}), \qquad \text{for } n = 3, 5... \tag{13}$$

So their matrix elements are

$$a_{n-1}^{BT} \to a_n, \qquad \text{for } n = 1, 3... \tag{14}$$

$$d_{n-1}^{BT} \to d_n, \qquad \text{for } n = 3, 5... \tag{15}$$

F. Dong (2008) [7]

Target mass corrections again. Ignoring the TMCs yields:

$$\begin{aligned}
\int dx x^{n-1} g_1(x, Q^2) &= g_1^{(n)} \\
&= a_n \\
&= \frac{1}{2}\tilde{a}_{n-1} \\
\int dx x^{n-1} g_2(x, Q^2) &= g_2^{(n)} \\
&= \frac{n-1}{n}\big(d_n - a_n\big) \\
&= \frac{n-1}{2n}\big(\tilde{d}_{n-1} - \tilde{a}_{n-1}\big)
\end{aligned}$$

for $n = 1, 3, 5\ldots$ and $n = 3, 5, \ldots$.

The Nachtmann moment $M_2^{(3)}$ in the massless limit is related to the CN moments:

$$I(Q^2) = \int x^2(2g_1 + 3g_2)dx = 2M_2^{(3)} \tag{16}$$

when $M^2/Q^2 \to 0$.

III. TARGET MASS CORRECTIONS

A. Twist Expansion

The twist expansion is

$$\begin{aligned} g_1^{(1)} &\equiv \int dxg1(x,Q^2) \\ &= \sum_{\tau=2,4,6...}^{\infty} \frac{\mu_\tau(Q^2)}{Q^{\tau-2}} \\ &= \mu_2 + \frac{\mu_4}{Q^2} + \mathcal{O}(\frac{1}{Q^4}) \end{aligned} \tag{17}$$

where τ is twist.

B. Cornwall-Norton Moments

Following Blumlein, Tkabladze (1999)[**?**], and Dong (2008)[7]

$$\begin{aligned} g_1^{(n)} &\equiv \int x^2 g_1(x,Q^2)dx \\ &= a_n + y^2 \frac{n(n+1)}{(n+2)^2}(na_{n+2} + 4d_{n+2}) \\ &\quad + y^4 \frac{n(n+1)(n+2)}{2(n+4)^2}(na_{n+4} + 8d_{n+4}) \\ &\quad + y^6 \frac{n(n+1)(n+2)(n+3)}{6(n+6)^2}(na_{n+6} + 12d_{n+6}) \\ &\quad + \mathcal{O}(y^8) \end{aligned} \tag{18}$$

$$\begin{aligned} g_2^{(n)} &\equiv \int x^2 g_2(x,Q^2)dx \\ &= \frac{n-1}{n}(d_n - a_n) \\ &\quad + y^2 \frac{n(n-1)}{(n+2)^2}(nd_{n+2} - (n+1)a_{n+2}) \\ &\quad + y^4 \frac{n(n-1)(n+1)}{2(n+4)^2}(nd_{n+4} - (n+2)a_{n+4}) \\ &\quad + y^6 \frac{n(n-1)(n+1)(n+2)}{6(n+6)^2}(nd_{n+6} - (n+3)d_{n+6}) \\ &\quad + \mathcal{O}(y^8) \end{aligned} \tag{19}$$

C. Nachtmann Moments

The Nachtmann variable [**?**]

At low Q^2, Nachtmann moments should be used instead of the CN moments. The Nachtmann moments are[4, 6]

$$\begin{aligned} M_1^{(n)}(Q^2) &\equiv a_n(Q^2) + \mathcal{O}\big(\frac{M^8}{Q^8}\big) \\ &= \frac{1}{2}\tilde{a}_{n-1}(Q^2) \\ &= \int_0^1 dx \frac{\xi^{n+1}}{x^2}\Big[\left(\frac{x}{\xi} - \frac{n^2}{(n+2)^2}\frac{M^2}{Q^2}x\xi\right) g_1(x,Q^2) - \frac{M^2}{Q^2}x^2\frac{4n}{n+2}g_2(x,Q^2)\Big] \end{aligned} \tag{20}$$

and

$$\begin{aligned} M_2^{(n)}(Q^2) &\equiv d_n(Q^2) + \mathcal{O}\big(\frac{M^8}{Q^8}\big) \\ &= \frac{1}{2}\tilde{d}_{n-1}(Q^2) \\ &= \int_0^1 dx \frac{\xi^{n+1}}{x^2}\Big[\frac{x}{\xi}g_1(x,Q^2) + \left(\frac{n}{n-1}\frac{x^2}{\xi^2} - \frac{n}{n+1}\frac{M^2}{Q^2}x^2\right) g_2(x,Q^2)\Big] . \end{aligned} \tag{21}$$

D. The Twist-3 Matrix Element

$$\begin{aligned} I(Q^2) &\equiv \int x^2\big(2g_1(x,Q^2) + 3g_2(x,Q^2)\big)dx \\ &= 2\Big[d_3 + 6\ d_5\frac{M^2}{Q^2} + 12\ d_7\frac{M^4}{Q^4} + 20\ d_9\frac{M^6}{Q^6}\Big] + \mathcal{O}\Big(\frac{M^8}{Q^8}\Big) \\ &= \Big[\tilde{d}_3 + 6\ \tilde{d}_5\frac{M^2}{Q^2} + 12\ \tilde{d}_7\frac{M^4}{Q^4} + 20\ \tilde{d}_9\frac{M^6}{Q^6}\Big] + \mathcal{O}\Big(\frac{M^8}{Q^8}\Big) \end{aligned} \tag{22}$$

E. Ellis-Jaffe Sum Rule

$$\Gamma_1 = \int_0^1 dx\ g_1(x) = \frac{1}{9}\Delta\Sigma \pm \frac{1}{12}a_3 + \frac{1}{36}a_8 \tag{23}$$

where the $\pm$ indicates proton or neutron, and we have introduced the moments of the flavor singlet distribution

$$\Delta\Sigma = \int_0^1 dx\ \Delta\Sigma(x)\ , \tag{24}$$

along with the non-singlet distributions

$$a_{3,8} = \int_0^1 dx\ \Delta q_{3,8}(x). \tag{25}$$

$$\int_0^1 dx g_1^p(x,Q^2) = C_{ns}(Q^2)\left(\pm\frac{1}{12}g_3(Q^2)+\frac{1}{36}a_8(Q^2)\right)+C_s(Q^2)\frac{1}{9}\Delta\Sigma(Q^2)+\mathrm{HT} \tag{26}$$

where C_{ns} and C_s are the non-singlet and the singlet coefficient functions respectively.

IV. MODEL CALCULATIONS

(1990)	Balitsky, et.al. [8]	Sum rules
(1995)	Stein, et.al. [9]	Sum rules
(1996)	Song [10] [9]	CM bag model
(1997)	Weigel, et.al.[11]	Chiral Soliton Model

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