Jefferson Lab
Personal note 06
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Error calculations of b_1^d

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From section 6 of Hoodbhoy, Jaffe and Manohar (Nuc. Phys. B312, p571-588, 1989), we have:

$$\frac{d\sigma_{\parallel}^{H}}{dxdy} = K\left[xF_1(x) + \left(\frac{2}{3} - H^2\right)xb_1(x)\right] \tag{1}$$

$$\frac{d\sigma_{\perp}^{H}}{dxdy} = K\left[xF_1(x) - \left(\frac{1}{3} - \frac{1}{2}H^2\right)xb_1(x)\right] \tag{2}$$

with $K = \frac{e^4 ME}{2\pi Q^4}[1 + (1-y)^2]$.

For simplicity, we will use $\sigma_{\parallel}$ for $\frac{d\sigma_{\parallel}^{H}}{dxdy}$ and $\sigma_{\perp}$ for $\frac{d\sigma_{\perp}^{H}}{dxdy}$. From Jaffe's email, we now know that $H^2 = (P+2)/3$, where P is the polarization of the target. Comparing $\frac{d\sigma_{\parallel}^{H}}{dxdy}$ with eq. 2.13 of Caroline Reidl's thesis, we can identify that $P = P_{zz}$ and it is the tensor polarization.

The tensor polarization can be extracted from P_z as follows:

$$P_{zz} = 2 - \sqrt{4 - 3P_z^2} \tag{3}$$

The tensor asymmetry A_{zz} depends on b_1 and F_1:

$$\frac{b_1}{F_1} = -\frac{3}{2}A_{zz} \tag{4}$$

The deuterium rates are estimated from the unpolarized deuteron cross section σ_D using MSTW2008:

$$R_D = \sigma_D \; dp \; d\Omega \; L = \sigma_D \; dp \; d\Omega \; \frac{I}{e} \; n_D \tag{5}$$

where L is the luminosity and the number of deuteron scattering centers in ammonia is:

$$n_D = 3 \cdot \mathcal{N_A} \cdot \frac{\rho_{ND_3}}{M_{ND_3}} \cdot pf \cdot z \tag{6}$$

where the factor 3 is for three deuterons in each ammonia molecule, $\mathcal{N_A}$ is the Avogadro's number, ρ_{ND_3} is the ammonia density (= 1.007 g/cm^3), M_{ND_3} is the ammonia molecular mass (= 20 g/mole), pf is the packing fraction (= 0.55) and z is target length (= 3 cm). The dilution factor f does not appear in this equation but used below.

To estimate the physics rates, the spectrometer acceptance and momentum bite were reduced to $d\Omega$ = 6.5 msr and $dp = \pm 8\%$ for the HMS and $d\Omega$ = 4.4 msr and $dp =^{+20\%}_{-8\%}$ for SHMS and a cut on $W \geq 1.8$ GeV was required.

1 Access to b_1 from the cross section difference

The deuteron target is diluted by the nitrogen contained in ND_3. So we can extract the deuteron contribution by substracting the nitrogen background $\sigma^U_{\parallel,\perp}$ as follows:

$$\begin{aligned} \sigma^T_\perp &= \sigma^D_\perp + \sigma^U_\perp \\ \sigma^T_\parallel &= \sigma^D_\parallel + \sigma^U_\parallel \end{aligned} \tag{7}$$

$$\tag{8}$$

Working with the equations of $\sigma_\parallel$ and $\sigma_\perp$, we can isolate b_1:

$$\sigma^T_\parallel - \sigma^T_\perp = \sigma^D_\parallel - \sigma^D_\perp = \frac{-K}{6}(2P^\parallel_{zz} + P^\perp_{zz})xb_1 \tag{9}$$

where $P^\parallel_{zz}$ and $P^\perp_{zz}$ are the tensor polarization achieved in the longitudinal and transverse configurations respectively. The unpolarized cross sections must be equal and therefore cancel. The expression of the unpolarized material cross sections can be simplified to: $\sigma^U_\perp = \sigma^U_\parallel = \sigma^U$.

In our measurement of b_1, we will get contributions from both the polarized and the unpolarized target materials, so the statistical uncertainty on b_1 should take into account the nitrogen background:

$$\begin{aligned}
\frac{\delta b_1}{b_1} &= \frac{\sqrt{(\delta\sigma_\perp^T)^2 + (\delta\sigma_\parallel^T)^2}}{\sigma_\perp^T - \sigma_\parallel^T} \\
&= \frac{\sqrt{(\delta\sigma_\perp^D)^2 + (\delta\sigma^U)^2 + (\delta\sigma_\parallel^D)^2 + (\delta\sigma^U)^2}}{(\sigma_\perp^D + \sigma^U) - (\sigma_\parallel^D + \sigma^U)} \\
&= \frac{\sqrt{(\delta\sigma_\perp^D)^2 + (\delta\sigma_\parallel^D)^2 + 2(\delta\sigma^U)^2}}{\sigma_\perp^D - \sigma_\parallel^D} \qquad (10)
\end{aligned}$$

In the valence region, $b_1 < 0$ which implies $\sigma_\perp^D < \sigma_\parallel^D$. In addition, b_1 is small and we can define the difference between $\sigma_\perp^D$ and $\sigma_\parallel^D$ as:

$$\sigma_\parallel^D = (1+\epsilon)\sigma_\perp^D \qquad \text{and} \qquad \delta\sigma_\parallel^{\mathrm{D}} = (1+\epsilon)\delta\sigma_\perp^{\mathrm{D}} \qquad (11)$$

Using these two relations in the expression of $\delta b_1/b_1$ and with 2ϵ and ϵ^2 negligible compared to 2, we obtain:

$$\begin{aligned}
\frac{\delta b_1}{b_1} &= \frac{\sqrt{[(1+\epsilon)^2 + 1](\delta\sigma_\perp^D)^2 + 2(\delta\sigma^U)^2}}{\sigma_\perp^D - (1+\epsilon)\sigma_\perp^D} \\
&= \frac{\sqrt{2(\delta\sigma_\perp^D)^2 + 2(\delta\sigma^U)^2}}{-\epsilon\sigma_\perp^D} \\
&= -\frac{\sqrt{2}}{\epsilon}\frac{\delta\sigma_\perp^T}{\sigma_\perp^D} \qquad (12)
\end{aligned}$$

We know that the cross sections are proportional to the counts $N_\perp^T$, $N_\perp^D$, etc, and the cross section errors to $\sqrt{N_\perp^T}$, $\sqrt{N_\perp^D}$, etc. So we write:

$$\frac{\delta b_1}{b_1} = -\frac{\sqrt{2}}{\epsilon}\frac{\sqrt{N_\perp^T}}{N_\perp^D} \qquad (13)$$

We also know that the dilution factor is the ratio of polarized to total cross sections, or polarized counts to total counts:

$$f(x,Q^2) = \frac{N_\perp^D(x,Q^2)}{N_\perp^T(x,Q^2})\propto \frac{\sigma_\perp^D(x,Q^2)}{\sigma_\perp^T(x,Q^2)} \qquad (14)$$

In the region of our measurement, the ratio of cross sections is 0.3 to a good approximation, given the equal numbers of protons and neutrons in D, ^{14}N and ^{4}He (the rates from He are suppressed by the packing fraction and $\rho_{He} << \rho_{ND_3}$).

Substituting $N_\perp^T = N_\perp^D/f$, we

$$\begin{aligned} \frac{\delta b_1}{b_1} &= -\frac{\sqrt{2}}{\epsilon}\frac{\sqrt{N_\perp^D/f}}{N_\perp^D} \\ &= -\frac{\sqrt{2}}{\epsilon}\frac{1}{\sqrt{fN_\perp^D}} \end{aligned} \tag{15}$$

$$N^D = \frac{2}{\epsilon^2 f(\delta b_1/b_1)^2} \tag{16}$$

The time necessary to achieve this statistics is:

$$T = \frac{N^D}{R_D} = \frac{2}{\epsilon^2 R_D f(\delta b_1/b_1)^2} \tag{17}$$

Similarly, substituting $N_\perp^D = f \cdot N_\perp^T$

$$\begin{aligned} \frac{\delta b_1}{b_1} &= -\frac{\sqrt{2}}{\epsilon}\frac{\sqrt{N_\perp^T}}{fN_\perp^T} \\ &= -\frac{\sqrt{2}}{\epsilon}\frac{1}{f\sqrt{N_\perp^T}} \end{aligned} \tag{18}$$

$$N_\perp^T = \frac{2}{\epsilon^2 (f\delta b_1/b_1)^2} \tag{19}$$

and in terms of the total rate $R_T = R_D/f$, so the expression of the time will be:

$$T = \frac{N_\perp^T}{R_T} = \frac{2}{\epsilon^2 R_T (f\delta b_1/b_1)^2} = \frac{2}{\epsilon^2 (R_D/f)(f\delta b_1/b_1)^2} \tag{20}$$

identical as Eq. 17, as expected.

1.1 Working with the relative error on b_1

Now to estimate the time necessary to perform a significant measurement of b_1 we need to have a idea of the value of ϵ.

$$\sigma_{||} = \sigma_u(1 - \frac{1}{3}P_{zz}^{||}\frac{b_1}{F_1}) \tag{21}$$

$$\sigma_{\perp} = \sigma_u(1 + \frac{1}{6}P_{zz}^{\perp}\frac{b_1}{F_1}) \tag{22}$$

with σ_u the unpolarized cross section.

$$1 + \epsilon = \frac{1 - \frac{1}{3}P_{zz}^{||}\frac{b_1}{F_1}}{1 + \frac{1}{6}P_{zz}^{\perp}\frac{b_1}{F_1}} \tag{23}$$

We use for b_1^d the fit from Kumano and for F_1^d MSTW2008 (no EMC effect or smearing included). For x-values of 0.30, 0.40 and 0.50, ϵ is equal to 0.00034, 0.00070 and 0.00099 respectively, assuming $P_{zz}^{\perp} = P_{zz}^{||} = 0.094$ for a vector polarization of $P_z = 0.35$.

1.2 Working with the absolute error on b_1

In order to extract the absolute uncertainty on b_1 and the time necessary to reach an absolute uncertainty δb_1, we multiply both sides Eq. 23 by $1 + \frac{1}{6}P_{zz}^{\perp}\frac{b_1}{F_1}$ and only use the approximation that $\epsilon << 1$. We obtain:

$$\frac{b_1}{\epsilon} = -\frac{6\ F_1}{2P_{zz}^{||} + P_{zz}^{\perp}} \tag{24}$$

The absolute error on b_1 has then the following expression:

$$\begin{aligned} \delta b_1 &= -\sqrt{2}\ \frac{b_1}{\epsilon}\ \frac{\delta\sigma_{\perp}^T}{\sigma_{\perp}^D} \\ &= \sqrt{2}\ \frac{6F_1}{2P_{zz}^{||} + P_{zz}^{\perp}}\ \frac{\delta\sigma_{\perp}^T}{\sigma_{\perp}^D} \\ &= \frac{\sqrt{2}}{f}\ \frac{6F_1}{2P_{zz}^{||} + P_{zz}^{\perp}}\ \frac{\delta\sigma_{\perp}^D}{\sigma_{\perp}^D} \end{aligned} \tag{25}$$

and the time necessary to obtain an absolute error δb_1 is:

$$T = \frac{2}{f\ R_D\ \delta b_1^2}\left(\frac{b_1}{\epsilon}\right)^2$$

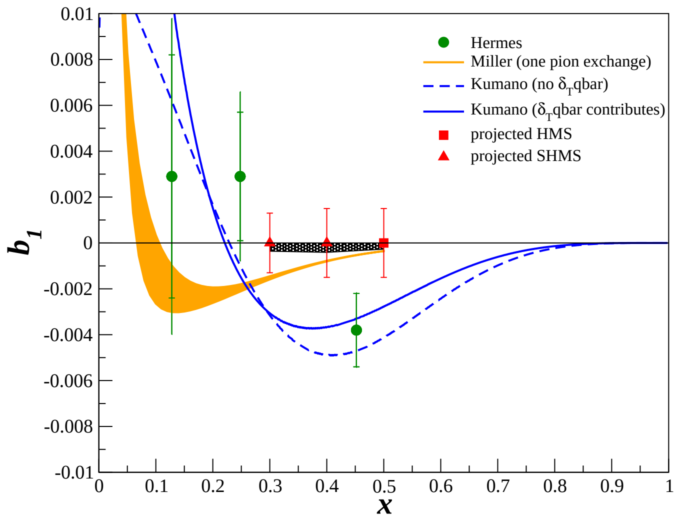

Figure 1: Projected statistical uncertainties for 28 days of production data with the new kinematics for the HMS (at 13.4°).

$$= \frac{72\ F_1^2}{f\ R_D\ (2P_{zz}^{\parallel} + P_{zz}^{\perp})^2\ \delta b_1^2} \tag{26}$$

Table 1: Updated kinematics, rates and projected uncertainties for b_1 and A_{zz}.

Spectro	$\bar{x}$	Q^2 (GeV^2)	$\bar{W}$ (GeV)	P_0 (GeV)	θ (deg.)	Rates (kHz)	δA_{zz} $\times 10^{-2}$	δb_1 $\times 10^{-2}$	time (days)
tensor polarization $P_{zz} = 9.4\%$									
SHMS	0.30	1.5	2.11	8.46	7.28	0.262	0.19	0.13	15.64
SHMS	0.40	2.2	2.07	8.20	8.96	0.079	0.40	0.15	12.41
HMS	0.50	4.1	2.25	6.83	13.42	0.007	0.87	0.15	28.00
tensor polarization $P_{zz} = 12.2\%$									
SHMS	0.30	1.5	2.11	8.46	7.28	0.262	0.15	0.10	15.50
SHMS	0.40	2.2	2.07	8.20	8.96	0.079	0.30	0.12	12.83
HMS	0.50	4.1	2.25	6.83	13.42	0.007	0.66	0.12	28.33

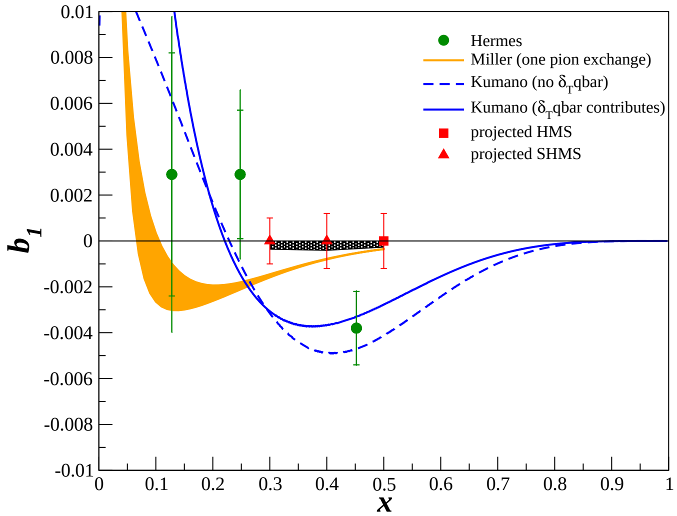

Figure 2: Projected statistical uncertainties for 28 days of production data with the new kinematics for the HMS (at 13.4°). An improvement of the tensor polarization of 2.8% absolute was assumed in this case.

1.3 Systematics

There is no contribution from the dilution factor, which only affects the statistics, not the systematics uncertainties, since we measure both σ_T and σ_D with the same systematics. Starting from Eq. 9 and the approximations of Eqs. 11, we have:

$$\begin{aligned} b_1 &= -\frac{6}{x\ K\ (2P^{\parallel}_{zz} + P^{\perp}_{zz})}(\sigma^D_{\parallel} - \sigma^D_{\perp}) \\ &= \frac{6\ \epsilon}{x\ K\ (2P^{\parallel}_{zz} + P^{\perp}_{zz})}\sigma^D_{\perp} \end{aligned} \tag{27}$$

Therefore, the relative systematics uncertainty on b_1 due to $\sigma^D_{\perp}$ is:

$$\left(\frac{\delta b_1}{b_1}\right)_{syst} = \left(\frac{\delta\sigma^D_{\perp}}{\sigma^D_{\perp}}\right)_{syst} \tag{28}$$

assuming that the $\sigma^D_{\perp}$ and $\sigma^D_{\parallel}$ have the same systematics.

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